Newtonian Dynamics

Newton's 2nd law of motion: $\sum_i \vec{f}_i = \frac{d\vec{p}}{dt}$ (momentum) — ALWAYS TRUE.

($\sum_i \vec{f}_i$: forces on a particle; $\frac{d\vec{p}}{dt}$: rate change of momentum with respect to time)

Newtonian? $\vec{p} = m\dot{\vec{r}}$ ← THIS MUST BE WITH RESPECT TO A FIXED OR INERTIAL POINT!

Then $\dot{\vec{p}} = \dot{m}\dot{\vec{r}} + m\ddot{\vec{r}}$. Often we'll say mass is a constant so $\dot{\vec{p}} = m\ddot{\vec{r}}$

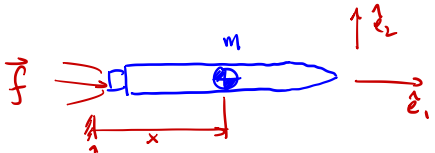
Anyway, $\sum_i \vec{f}_i = \dot{m}\dot{\vec{r}} + m\ddot{\vec{r}}$ OR $m\ddot{\vec{r}}$.

- $\dot{m}\dot{\vec{r}}$: IF YOU HAVE THIS IN YOUR PROBLEM, IT CAN BE A HASSLE. See "Ito Calculus" FOR MORE.
- $m\ddot{\vec{r}}$: WE'VE DONE THIS: $\vec{r} \to \dot{\vec{r}} \to \ddot{\vec{r}}$
- $\dot{m}\dot{\vec{r}}$: UM, HAVEN'T DONE THIS...

FINDING $\sum_i \vec{f}_i$ ... usually takes a FREE-BODY DIAGRAM.

So... that's it. That's Newtonian dynamics.

EXAMPLE: Rocket.



THE LINES HERE SUGGEST IT'S INERTIAL...

$\underbrace{\sum \vec{f}_i}_{①} = \dot{\vec{p}} = \underbrace{\dot{m}\dot{\vec{r}} + m\ddot{\vec{r}}}_{②}$

① $\vec{f} = f\hat{e}_1$

② POSITION VECTOR: $\vec{r} = x\hat{e}_1$

→ VEL VECTOR: $\dot{\vec{r}} = \dot{x}\hat{e}_1 + x\dot{\hat{e}}_1$ (ZERO BASED ON INSPECTION: $\vec{f}$ and $\dot{\vec{r}}$ are parallel)

→ $\ddot{\vec{r}} = \ddot{x}\hat{e}_1$

Substituting in for ① & ②:

$$\sum \vec{f}_i = f\hat{e}_1 = \dot{m}\dot{x}\hat{e}_1 + m\ddot{x}\hat{e}_1 = (\dot{m}\dot{x} + m\ddot{x})\hat{e}_1.$$

Look along $\hat{e}_1$: $\hat{e}_1 \cdot \sum_i \vec{f}_i = f = \hat{e}_1 \cdot (\dot{m}\dot{x} + m\ddot{x})\hat{e}_1 = \dot{m}\dot{x} + m\ddot{x} \Rightarrow f = \dot{m}\dot{x} + m\ddot{x}$ along $\hat{e}_1$.

Along $\hat{e}_2$? $\hat{e}_2 \cdot \sum_i \vec{f}_i = 0 = \hat{e}_2 \cdot (\dot{m}\dot{x} + m\ddot{x})\hat{e}_1 = 0$

So, not much more we can do here... unless we have more info. about $f = f(t,x)$.

FIRST: SIMPLE CASE: $f = f_0$ constant; mass ~ constant

$$f = \dot{m}\dot{x} + m\ddot{x} \Rightarrow f_0 = m\ddot{x} \Rightarrow \ddot{x} = f_0/m \Rightarrow \dot{x} = \left(\frac{f_0}{m}\right)t + c_1 \rightarrow$$

c_1 ↳ const.

$$x = \frac{f_0}{2m}t^2 + c_1 t + c_2$$

SECOND: Suppose $m \sim$ constant for our rocket but $f(t) = \frac{f_0}{2}\left[1 + \sin\beta t\right]$

$$\frac{f_0}{2}\left[1+\sin\beta t\right] = m\ddot{x} \rightarrow \ddot{x} = \frac{f_0}{2m}\left[1+\sin\beta t\right] \rightarrow \dot{x} = \frac{f_0}{2m}\left[t - \frac{1}{\beta}\cos\beta t + \frac{1}{\beta}\right]$$

IF we assume $\dot{x}(t=0) = 0$.

p35: EX 3.3.5: $f = f(x)$.

Linear Momentum and the Impulse

$\sum_i \vec{f}_i = \dot{\vec{p}}$ → INTEGRATE OVER TIME: $\int_{t_1}^{t_2} \vec{F}\,dt = \int_{t_1}^{t_2} \dot{\vec{p}}\,dt$

$$\rightarrow \int_{t_1}^{t_2} \dot{\vec{p}}\,dt = \vec{p}_2 - \vec{p}_1 \rightarrow$$

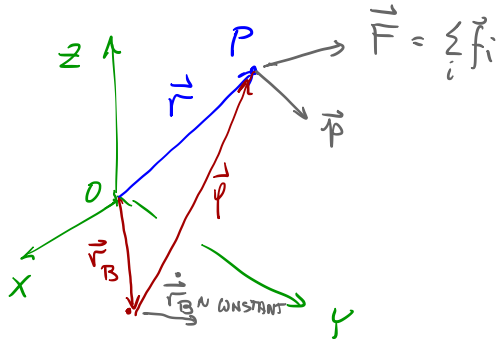
$$\hat{\vec{F}} \equiv \int_{t_1}^{t_2} \left(\sum_i \vec{f}_i\right) dt = \vec{p}_2 - \vec{p}_1 \rightarrow \vec{p}_2 = \vec{p}_1 + \hat{\vec{F}}$$

$\hat{\vec{F}}$ ↳ IMPULSE

$\vec{p}_2 - \vec{p}_1$ CHANGE IN MOMENTUM

Angular Momentum

So we have $\sum_i \vec{f}_i = \dot{\vec{p}}$



B ~ external point

Let's take $\vec{\rho}\times$ to both sides: $\vec{\rho} \times \left(\sum_i \vec{f}_i\right) = \vec{\rho} \times \dot{\vec{p}}$

$\vec{\rho} \times \dot{\vec{p}}$ => Let's look at $\frac{d}{dt}\left(\vec{\rho}\times\vec{p}\right)$ instead: $\frac{d}{dt}\left(\vec{\rho}\times\vec{p}\right) = \dot{\vec{\rho}}\times\vec{p} + \vec{\rho}\times\dot{\vec{p}}$ =>

$$\vec{\rho}\times\dot{\vec{p}} = \frac{d}{dt}\left(\vec{\rho}\times\vec{p}\right) - \dot{\vec{\rho}}\times\vec{p}$$

... eventually: $\vec{M}_B = \frac{d}{dt}\left(\vec{H}_B\right)$